

THE MOTIVE OF THE FANO SURFACE OF LINES

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ABSTRACT. In this short note, we prove that the Chow motive of the Fano surface of lines on the smooth cubic threefold is finite-dimensional in the sense of Kimura. This gives an example of a variety not dominated by a product of curves whose Chow motive is of Abelian type.

1. INTRODUCTION

Let k be a field with $\text{char } k \neq 2$ and let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. We denote by $S(X)$ (or just S) the Fano variety of lines in X . This is known to be a smooth, connected projective surface of general type. It turns out that this surface possesses a great many remarkable properties. It is known, for instance, that the Albanese map is an imbedding $i : S \hookrightarrow A := \text{Alb}(S)$ and that the pull-back $H^2(A) \xrightarrow{i^*} H^2(S)$ is an isomorphism. In this note, we will prove a motivic version of this isomorphism. For this, let \mathcal{M}_k denote the category of pure Chow motives with rational coefficients and let $\mathfrak{h} : \mathcal{V}_k^{\text{opp}} \rightarrow \mathcal{M}_k$ be the functor that sends a smooth projective k -variety X to $\mathfrak{h}(X) = (X, \Delta_X, 0) \in \mathcal{M}_k$. Then, our first result is that:

Theorem 1.1. *The morphism $\mathfrak{h}(i) : \mathfrak{h}(A) \rightarrow \mathfrak{h}(S)$ is split-surjective.*

This will show that the motive of the Fano surface is *finite-dimensional in the sense of Kimura*. The primary known examples of surfaces with finite-dimensional motive are those for which either:

- (i) The Chow group of nullhomologous 0-cycles is representable.
- (ii) There exists a dominant rational map from a product of smooth projective curves.

Since $p_g(S) > 0$, Mumford's Theorem ([15] Theorem 3.13) shows that $CH_{\text{hom}}^2(S)$ is not representable (when $k = \mathbb{C}$). Moreover, a recent result in [14] shows that S is not dominated by a product of curves. To the author's knowledge, the Fano surface is the first example of a surface with finite-dimensional motive for which neither (i) nor (ii) holds. One reason for interest in finite-dimensionality is that if $M \in \mathcal{M}_k$ is finite-dimensional, then any morphism $f : M \rightarrow M$ that induces an isomorphism on cohomology is actually an isomorphism of motives. This will be important to proving:

Theorem 1.2. *The pull-back along the Albanese imbedding $i : S \hookrightarrow A$ induces an isomorphism $\mathfrak{h}^2(A) \xrightarrow{\cong} \mathfrak{h}^2(S)$ in \mathcal{M}_k .*

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The plan will be as follows. We will give a proof of the Theorem 1.1, then review the notion of finite-dimensionality for motives which will allow us to prove Theorem 1.2. In this note, all Chow groups will be taken with rational coefficients.

2. A GENERAL PRINCIPLE

The following variant of the Manin principle will facilitate the proof of Theorem 1.1, as pointed out by B. Kahn.

Proposition 2.1. *Let X and Y be smooth projective connected varieties over a field k with a morphism $f : Y \rightarrow X$ such that $f_\Omega^* : CH^i(X_\Omega) \rightarrow CH^i(Y_\Omega)$ is surjective for all i and every algebraically closed extension $k \subset \Omega$. Then, $\mathfrak{h}(f) : \mathfrak{h}(X) \rightarrow \mathfrak{h}(Y)$ is split-surjective.*

This principle has appeared in the literature in various guises (see, for instance, [7] Theorems 3.5 and 3.6). However, since this exact statement is difficult to find, we give a proof below, following [8]. We subdivide the proof into three claims:

Claim: $f_L^* : CH^i(X_L) \rightarrow CH^i(Y_L)$ is surjective for all i and every extension $k \subset L$.

Proof of Claim. Let $\beta \in CH^i(Y_L)$. We need to find $\alpha \in CH^i(X_L)$ for which $f_L^*(\alpha) = \beta$. By assumption, $f_{\overline{L}}^* : CH^i(X_{\overline{L}}) \rightarrow CH^i(Y_{\overline{L}})$ is surjective, so there is some $\overline{\alpha} \in CH^i(X_{\overline{L}})$ such that $f_{\overline{L}}^*(\overline{\alpha}) = \beta_{\overline{L}}$. Let $L' \supset L$ be some finite Galois extension for which there is $\alpha' \in CH^i(X_{L'})$ with $\alpha'_{\overline{L}} = \overline{\alpha}$. Indeed, there is some finite extension for which this holds; since Chow groups are unchanged by passing to a purely inseparable extension, we can assume that L' is a separable extension. This allows us to pass to a Galois extension. Then, let $G = \text{Gal}(L'/L)$ and define $\text{Tr}(\alpha') = \frac{1}{|G|} \sum_{g \in G} g^* \alpha'$. By [5] Example 1.7.2.6, this yields $\alpha \in CH^i(X_L)$ such that $\alpha_{L'} = \text{Tr}(\alpha')$. We compute:

$$(f_L^*(\alpha))_{L'} = f_{L'}^*(\alpha_{L'}) = f_{L'}^*(\text{Tr}(\alpha')) = \text{Tr}(f_{L'}^*(\alpha')) = \text{Tr}(\beta_{L'}) = \beta_{L'}$$

Since $CH^i(X_L) \rightarrow CH^i(X_{L'})$ is injective, it follows that $f_L^*(\alpha) = \beta$, as desired. \square

Claim: $(f \times \text{id}_Z)^* : CH^i(X \times Z) \rightarrow CH^i(Y \times Z)$ is surjective for all i and all projective varieties Z over k .

Proof of Claim. One easily reduces to the case that Z is connected. We first observe that the pull-back

$$(f \times \text{id}_Z)^* : CH^i(X \times Z) \rightarrow CH^i(Y \times Z)$$

is defined even when Z is singular. Indeed, f is a local complete intersection morphism since the source and the target are smooth, and one readily checks that so is the (fiber) product

$$f \times \text{id}_Z : X \times Z \rightarrow Y \times Z.$$

Thus, using [5] Chapter 6.6, we can define $(f \times id_Z)^*$. We then observe the following commutative diagram with rows exact:

$$(1) \quad \begin{array}{ccccccc} \varinjlim_W CH^{i-c}(X \times W) & \xrightarrow{(id_X \times j_W)^*} & CH^i(X \times Z) & \longrightarrow & CH^i(X_{k(Z)}) & \longrightarrow & 0 \\ (f \times id_W)^* \downarrow & & (f \times id_Z)^* \downarrow & & (f_{k(Z)})^* \downarrow & & \\ \varinjlim_W CH^{i-c}(Y \times W) & \xrightarrow{(id_Y \times j_W)^*} & CH^i(Y \times Z) & \longrightarrow & CH^i(Y_{k(Z)}) & \longrightarrow & 0 \end{array}$$

where $k(Z)$ is the function field of Z , the limit ranges over subvarieties $j_W : W \hookrightarrow Z$ with $\dim(W) \leq \dim(Z)$ and $c = \dim(Z) - \dim(W)$. The localization sequence implies that the rows are exact. For the commutativity of the left diagram, note first that there is a Cartesian diagram:

$$(2) \quad \begin{array}{ccc} Y \times W & \xrightarrow{id_Y \times j_W} & Y \times Z \\ f \times id_W \downarrow & & f \times id_Z \downarrow \\ X \times W & \xrightarrow{id_X \times j_W} & X \times Z \end{array}$$

with the left and right vertical arrows of the same relative codimension. Then, using [5] Theorem 6.2, one obtains the desired commutativity.

The claim then follows by an induction argument on $n = \dim(Z)$. The case $n = 0$ follows from the previous claim. Assume then that it holds for $n - 1$. Then, we note that the rightmost vertical arrow in (1) is surjective by the previous claim. The leftmost arrow is surjective by the inductive hypothesis. A diagram chase then shows that the statement is true for n . Hence, the claim. \square

Claim: $\mathfrak{h}(f)$ possesses a right-inverse.

Proof of Claim. Taking $Z = Y$ and $i = \dim(Y)$ in the above claim, we obtain that

$$CH^i(Y \times X) \xrightarrow{(id_Y \times f)^*} CH^i(Y \times Y)$$

is surjective. So, there is some $\gamma \in CH^i(X \times Y)$ for which $(id_Y \times f)^* \gamma = \Delta_Y$. Applying Lieberman's lemma ([5] Proposition 16.1.1) then gives

$$\mathfrak{h}(f) \circ \gamma = (id_Y \times f)^* \gamma = \Delta.$$

This is the desired result. \square

3. PROOF OF THEOREM 1.1

Assume that S has a k -rational point and let $i : S \rightarrow A$ be the corresponding Albanese morphism. To prove the theorem, we need to show that the correspondence ${}^t\Gamma_i \in \text{Cor}^0(A \times S)$ possesses a right-inverse. The general principle then shows that it suffices to prove that the pull-back $i_\Omega^* : CH^j(A_\Omega) \rightarrow CH^j(S_\Omega)$ is surjective for $j = 1, 2$ and all algebraically closed extensions $k \subset \Omega$.

For a smooth cubic threefold X over k , S is the Hilbert scheme of X with Hilbert polynomial $t + 1$. We recall the following base change compatibility for Hilbert schemes (see, for instance, [12]); i.e., for an extension $k \subset \Omega$, we have

$$S(X_\Omega) \cong S(X)_\Omega$$

Thus, S_Ω is the Fano surface of lines of X_Ω . Moreover, since the Albanese is compatible with base extension, we view $i_\Omega : S_\Omega \rightarrow A_\Omega$ as an Albanese map.

Proposition 3.1. *For all algebraically closed extensions $k \subset \Omega$, the pull-back $i_\Omega^* : CH^1(A_\Omega) \rightarrow CH^1(S_\Omega)$ is an isomorphism.*

Proof. Since $i_\Omega : S_\Omega \rightarrow A_\Omega$ is an Albanese map, we have $Pic^0(A_\Omega) \xrightarrow{i_\Omega^*} Pic^0(S_\Omega)$ is an isomorphism, and so it suffices to show that we have an isomorphism $i_\Omega^* : NS(A_\Omega)_\mathbb{Q} \rightarrow NS(S_\Omega)_\mathbb{Q}$ of Néron-Severi groups. Now, let $\ell \neq \text{char } k$. Then, $H^2(A_\Omega, \mathbb{Q}_\ell(1)) \xrightarrow{i_\Omega^*} H^2(S_\Omega, \mathbb{Q}_\ell(1))$ is an isomorphism by [13] Proposition 4. Note that S and A may be defined over some finitely generated field k_0 . Upon passing to a large enough extension of k_0 , we may also assume that C possesses a model over k_0 and that $NS(C_{k_0} \times A_{k_0})_\mathbb{Q} \cong NS(C \times A)_\mathbb{Q}$ (and, similarly, for $C \times S$). Note that this is possible because the Néron-Severi group is finitely generated. Thus, we need to show that

$$NS(C_{k_0} \times A_{k_0})_\mathbb{Q} \xrightarrow[\cong]{(id_C \times i)^*} NS(C_{k_0} \times S_{k_0})_\mathbb{Q}$$

For this, let $G := \text{Gal}(k/k_0)$ be the absolute Galois group and $\ell \neq \text{char } k$. The Künneth theorem on cohomology then gives a (G -module) isomorphism:

$$(3) \quad H^2(C \times A, \mathbb{Q}_\ell(1)) \xrightarrow[\cong]{(id_C \times i)^*} H^2(C \times S, \mathbb{Q}_\ell(1))$$

The result then follows from applying the functor $H^0(G, -)$ to (3) and noting that the Tate conjecture holds for the left-hand side (by Faltings' theorem and the fact that k_0 is a finitely generated field). \square

We have the following important result of Bloch:

Proposition 3.2 ([3] 1.7). *Let S be the Fano surface of lines of a smooth cubic threefold in \mathbb{P}^4 over an algebraically closed field of characteristic $\neq 2$. Then, the intersection product $CH^1(S) \otimes CH^1(S) \xrightarrow{\cdot} CH^2(S)$ is surjective.*

Proposition 3.3. *For all algebraically closed extensions $k \subset \Omega$, the pullback $CH^2(A_\Omega) \xrightarrow{i_\Omega^*} CH^2(S_\Omega)$ is surjective.*

Proof. Since pull-back commutes with intersection product on Chow groups, we have the following diagram:

$$(4) \quad \begin{array}{ccc} CH^1(A_\Omega) \otimes CH^1(A_\Omega) & \xrightarrow{\cdot} & CH^2(A_\Omega) \\ i_\Omega^* \times i_\Omega^* \downarrow & & i_\Omega^* \downarrow \\ CH^1(S_\Omega) \otimes CH^1(S_\Omega) & \xrightarrow{\cdot} & CH^2(S_\Omega) \end{array}$$

From Proposition 3.1, the left vertical arrow is surjective. Since Ω is algebraically closed and S_Ω is the Fano surface of lines of X_Ω , we can apply Proposition 3.2 to deduce that the bottom horizontal arrow is also surjective.. So, the right vertical arrow is surjective, as desired. \square

4. FINITE-DIMENSIONALITY OF MOTIVES

This section reviews the definition and properties of finite-dimensionality before proving Theorem 1.2. Recall that the category of Chow motives \mathcal{M}_k is a tensor category with tensor product defined as:

$$(5) \quad (X, \pi, m) \otimes (Y, \tau, n) := (X \times Y, \pi \times \tau, m + n)$$

We can define an action of the symmetric group $\mathbb{Q}[\mathfrak{S}_n] \rightarrow \text{End}_{\mathcal{M}_k}(M^{\otimes n})$ for $M \in \mathcal{M}_k$. Since \mathcal{M}_k is a pseudo-Abelian category, all idempotents possess images in \mathcal{M}_k . So, for any idempotent in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$, there is a corresponding motive. In particular, we have

$$(6) \quad \begin{aligned} \text{Sym}^n M &= \text{Im}(\pi_{\text{sym}}) \\ \wedge^n M &= \text{Im}(\pi_{\text{alt}}) \end{aligned}$$

for the symmetric and the alternating representation of \mathfrak{S}_n .

Definition 4.1 ((Kimura)). *A motive $M \in \mathcal{M}_k$ is said to be oddly finite-dimensional if $\text{Sym}^n M = 0$ for $n \gg 0$ and evenly finite-dimensional if $\wedge^n M = 0$ for $n \gg 0$. M is said to be finite-dimensional if $M = M_+ \oplus M_-$, where M_+ is evenly finite-dimensional and M_- is oddly finite-dimensional.*

We have the following properties of finite-dimensional motives:

Theorem 4.1 ((Kimura, [9])).

- (a) *The motive of a smooth projective curve is finite-dimensional.*
- (b) *If $M, N \in \mathcal{M}_k$ are finite-dimensional, then so are $M \oplus N$ and $M \otimes N$. Conversely, if $M \oplus N$ is finite-dimensional, then so are M and N .*
- (c) *If $f : M \rightarrow N$ is split-surjective and M is finite-dimensional, then so is N .*
- (d) *If M is finite-dimensional and the odd degree cohomology $H^-(M) = 0$ for some Weil cohomology H^* , then M is evenly finite-dimensional (and, similarly for oddly finite-dimensionality).*
- (e) *Suppose M is finite-dimensional and $\Phi \in \text{End}_{\mathcal{M}_k}(M)$ is such that $\Phi_* \in \text{End}(H^*(M))$ is an isomorphism. Then, Φ is an isomorphism.*

As a consequence of (a), (b), the motive of any product of smooth projective curves is finite-dimensional; from (c), so is any variety dominated by a product of curves (such as Abelian varieties). In [6] it is demonstrated that varieties of dimension ≤ 3 have finite-dimensional motive if $CH_0(X)_{\text{hom}}$ is representable. (This is true, in particular, if X is a rationally connected threefold.) Other than this, the following conjecture remains wide open.

Conjecture 4.1 ((Kimura, O’Sullivan)). *Every motive $M \in \mathcal{M}_k$ is finite-dimensional.*

To prove Theorem 1.2, we will need the next two results.

Proposition 4.1 ([11] Theorem 3). *Let X be a smooth projective surface. Then, there are idempotents $\pi_i \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}(X))$ satisfying the following conditions:*

- (i) $\Delta_X = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4$
- (ii) $\pi_i \circ \pi_j = 0$ for $i \neq j$
- (iii) *The motive $\mathfrak{h}^i(X) = (X, \pi_i, 0)$ satisfies $H^*(\mathfrak{h}^i(X)) = H^i(X)$ for any Weil cohomology H^* .*

Remark 4.1. *It should be noted that the idempotents in Proposition 4.1 are not unique. Some extra conditions (see loc. cit.) may be imposed on π_i so that the resulting motives are unique up to isomorphism.*

When X is an Abelian variety of dimension g , we have the following result of Denninger and Murre which was proved using Beauville's Fourier transform ([2]).

Theorem 4.2 ([4] Theorem 3.1)). *There is a unique decomposition*

$$(7) \quad \mathfrak{h}(X) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(X)$$

where $\mathfrak{h}^i(X) = (X, \pi_i, 0)$ with π_i idempotents satisfying:

- (i) $\pi_i \circ \pi_j = 0$ for $i \neq j$;
- (ii) ${}^t\Gamma_n \circ \pi_i = n^i \cdot \pi_i = \pi_i \circ {}^t\Gamma_n$ for all $n \in \mathbb{Z}$;
- (iii) ${}^t\pi_i = \pi_{2g-i}$.

Remark 4.2. *As suggested by the notation, we have $H^*(\mathfrak{h}^i(X)) = H^i(X)$.*

Since the motive of an Abelian variety X is finite-dimensional and the cohomology of $\mathfrak{h}^i(X)$ is concentrated in one degree, Theorem 4.1 implies that $\mathfrak{h}^i(X)$ is finite-dimensional of parity $i \pmod{2}$. From Theorem 1.1 and Theorem 4.1 (c), it follows that the motive of the Fano surface $\mathfrak{h}(S)$ is also finite-dimensional and the summand $\mathfrak{h}^2(S)$ as in Proposition 4.1 is evenly finite-dimensional.

Proof of Theorem 1.2. Fix $\pi_{2,S}$ as in Proposition 4.1. We define

$$\mathfrak{h}^2(i) = \pi_{2,S} \circ {}^t\Gamma_i \circ \pi_{2,A} \in \text{Hom}_{\mathcal{M}_k}(\mathfrak{h}^2(A), \mathfrak{h}^2(S))$$

The goal is to show that $\mathfrak{h}^2(i)$ is an isomorphism of motives. This means that we need to find some $\psi \in \text{Hom}_{\mathcal{M}_k}(\mathfrak{h}^2(S), \mathfrak{h}^2(A))$ for which

$$(8) \quad \psi \circ \mathfrak{h}^2(i) = \pi_{2,A} \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}^2(A)), \quad \mathfrak{h}^2(i) \circ \psi = \pi_{2,S} \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}^2(S))$$

Since $\mathfrak{h}^2(A)$ and $\mathfrak{h}^2(S)$ are evenly finite-dimensional, it suffices by Theorem 4.1 (e) to find some such ψ for which these equalities hold cohomologically. Moreover, $i^* = \mathfrak{h}^2(i)_* : H^2(A, \mathbb{Q}_\ell) \rightarrow H^2(S, \mathbb{Q}_\ell)$ is an isomorphism for $\ell \neq \text{char } k$. Thus, if we can find some

$$(9) \quad \gamma \in CH^2(S \times A)$$

for which $\gamma_* : H^2(S, \mathbb{Q}_\ell) \rightarrow H^2(A, \mathbb{Q}_\ell)$ is the inverse of i^* , we can set $\psi := \pi_{2,A} \circ \gamma \circ \pi_{2,S}$, and this gives the desired correspondence in (8).

From [1], the image of the map $S \times S \rightarrow A$ defined by $(x, y) \mapsto i(x) - i(y)$ is an ample divisor Θ . Moreover, we have

$$\frac{1}{3!} \cdot \Theta \wedge \Theta \wedge \Theta = [S] \in H^6(A, \mathbb{Q}_\ell(3))$$

The Hard Lefschetz theorem then implies that $\wedge[S] : H^2(A, \mathbb{Q}_\ell) \rightarrow H^8(A, \mathbb{Q}_\ell)(3)$ is an isomorphism. Also, since the Lefschetz standard conjecture is true for Abelian varieties ([10] Proposition 4.3), it follows that there is a correspondence $\phi \in CH^2(A \times A)$ such that $\phi_* : H^8(A, \mathbb{Q}_\ell) \rightarrow H^2(A, \mathbb{Q}_\ell)(-3)$ is the inverse of $\wedge[S]$. The projection formula then shows that $i_* \circ i^* = \wedge[S]$ so that $\phi_* \circ i_* \circ i^*$ is the identity on $H^2(A, \mathbb{Q}_\ell)$. Since $i^* : H^2(A, \mathbb{Q}_\ell) \rightarrow H^2(S, \mathbb{Q}_\ell)$ is an isomorphism, it follows that $\gamma = \phi \circ \Gamma_i$ is the desired inverse. □

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